Statistical accuracy in the digital autocorrelation of photon counting fluctuations

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above threshold. Quantitative comparison of such data with theory is always performed through computation of the factorial moments since these functions are independent of beam attenuation (Pike 1969).

The dependence of the second moment of the intensity fluctuation $\langle I^2 \rangle$ upon integration time can be very accurately predicted by use of the following approximate formula:

$$\frac{\langle I^2 \rangle}{\langle I \rangle^2} = 1 + \left\{ \pi \left(\frac{I_0}{I} \right)^2 \left(\frac{p}{\sqrt{\pi}} \frac{I}{I_0} + 1 \right) - 2 \right\} \left(\frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2} \right).$$

This is simply constructed by assuming the intensity fluctuation spectrum to be a single Lorentzian line and using the formula (Jakeman and Pike 1969)

$$\frac{d^2}{dT^2} \left(T^2 \frac{\langle I^2 \rangle}{\langle I \rangle^2} \right) = 2(1 + C e^{-s})$$

together with the known form of $\langle I^2 \rangle / \langle I \rangle^2$ at $T = 0$ which defines C.

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Statistical accuracy in the digital autocorrelation of photon counting fluctuations

Abstract. Optical measurements by intensity fluctuation spectroscopy are subject to errors arising from the statistical nature of light and of the photodetection process. We report here the results of a calculation of the expected errors, due to these causes, in linewidth measurements by digital autocorrelation of photon counting fluctuations.

As the methods of intensity fluctuation spectroscopy are applied to a greater variety of problems (see for example Benedek 1968, Pike 1969, Cummins and Swinney 1970) it becomes increasingly important to assess their accuracy as a function of the

experimental variables. Not only would this make it possible to predict the accuracy of a measurement obtained with a given experimental arrangement, but the use of expensive equipment could also be optimized for a given purpose. The problem has been investigated in the special case corresponding experimentally to using a detector area much larger than a coherence area of the light by several authors (Benedek 1968, Haus 1969, Cummins and Swinney 1970). More recently, Pusey (private communication) has evaluated the error in the intensity autocorrelation function in the limit of zero time delay when the light is sampled over periods which are short compared with its coherence time. Also making the latter assumption, Degiorgio and Lastovka (private communication) have calculated the variance of the intensity autocorrelation function of Gaussian–Lorentzian light for arbitrary delay times. In addition they have computed the error in the linewidth, obtained by a three-parameter least-squares fitting procedure, for this case.

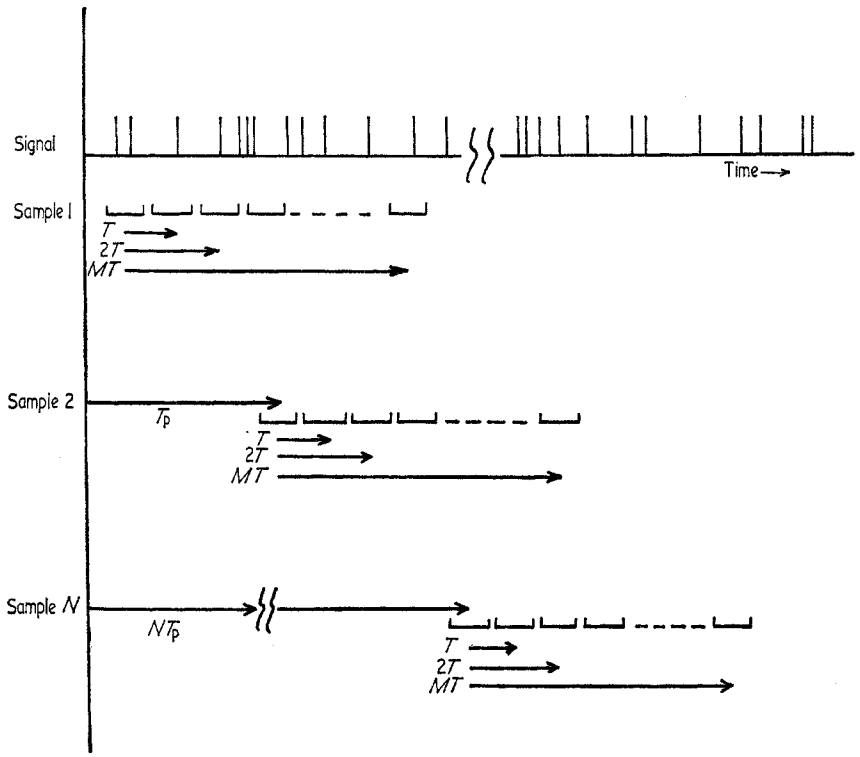

Figure 1. Sampling scheme used by Foord *et al.* (1970).

We have independently calculated the expected error in the intensity autocorrelation function and in the linewidth (obtained by a two-parameter least-squares fitting appropriate to our own experimental procedure) of Gaussian–Lorentzian light, for the more general case of arbitrary sample time. We present here some of the more important results of this calculation, taken from a complete treatment of the problem to appear in due course.

We consider the experimental situation described by Foord *et al.* (1970) which uses the sampling scheme shown in figure 1. Samples of the autocorrelation function are constructed at intervals T_p, each sample consisting of M channels containing the

product $n(t; T)\, n(t+rT; T)$ where the integer r runs from 1 to M, and $n(t; T)$ is the number of photons counted in the interval T centred at time t. The sample time T is the resolution time of the instrument if, as we shall assume, dead-time effects are negligible. In the experiments referred to above all M delayed products were constructed within the time T and sampling was carried out with a period $T_p = T$ to minimize information wastage. Drift of the mean photoelectron count rate during the course of a series of experiments necessitates normalization of each complete measurement and the quantity determined experimentally, namely

$$\hat{g}^{(2)}(\tau; T) = \frac{N^{-1} \sum_{r=1}^{N} n(rT+\tau; T) n(rT; T)}{\left\{ N^{-1} \sum_{r=1}^{N} n(rT; T) \right\}^2} \tag{1}$$

where τ is the delay time, and N is the number of samples taken, is a biased estimate of the normalized intensity autocorrelation function. We have evaluated analytically the variance of the unbiased estimator

$$\hat{G}^{(2)}(\tau; T) = N^{-1} \sum_{r=1}^{N} n(rT+\tau; T) n(rT; T) \tag{2}$$

of the un-normalized intensity correlation function for the case of Gaussian-Lorentzian light, using the methods of Jakeman and Pike (1968) and the finite-sample-time results of Jakeman (1970). The variance of the biased estimator (1) is more difficult to calculate and we have retained terms of order N^{-1} only. This will normally be a very good approximation and leads to a relatively simple analytic form for the error in $g^{(2)}(\tau; T)$. Experimentally, an estimate $\hat{\Gamma}$ for the optical linewidth is obtained by a least-squares fit of the measured values of (1) to the function

$$1 + C \exp(-2\hat{\Gamma}\tau). \tag{3}$$

C depends on the sample time, linewidth, detector area and dead-time effects and, although it can sometimes be calculated, is normally treated as an independent parameter. We have investigated the weighting to be used in the above procedure and have expressed the variance of Γ for the optimum distribution in terms of the variance of $\hat{g}^{(2)}(\tau; T)$.

The results obtained for the percentage error in Γ in the case of Gaussian–Lorentzian light are plotted as a function of ΓT for various values of r (the mean number of counts per coherence time) and M in figure 2. The curves relate to an experiment in which $N\Gamma T$ is 10^4. At such typical large values the ordinate scales as $1/(N\Gamma T)^{1/2}$. For a fixed number of channels there is an optimum choice of sample time defined roughly by $M\Gamma T \sim 1$ to 3 according to the value of r. There is nothing to be gained by increasing M beyond the value for which the optimum choice of T is such that $r\Gamma T \sim 10^{-2}$. Moreover, provided that $\Gamma T > 10^{-2}$ the error cannot be reduced by increasing the number of counts per coherence time beyond about ten. If $r > 10$ it is advantageous to reduce the detector area until r is of this order since this will reduce spatial averaging effects (Jakeman *et al.* 1970). When the statistics of the detection process dominate the error ($r\Gamma T \ll 1$) it is possible to improve the situation by increasing the detector area up to about a coherence area provided that r remains less than 10. Little further can be gained in any circumstance by increasing the detector area beyond this size.

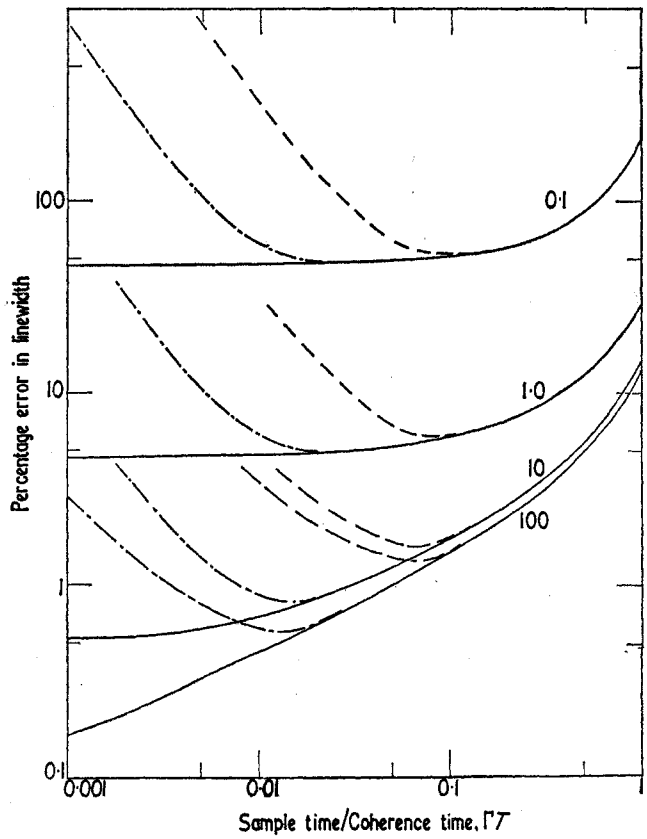

Figure 2. Percentage error in linewidth for single Gaussian–Lorentzian line. $N\Gamma T = 10^4$. Each set of curves is labelled with the number of counts per coherence time. ——, — · —, — — — refer to an infinite number, one hundred, and twenty channels respectively.

In a number of recent experiments the autocorrelation function has been found by using a 'clipping' technique (Jakeman and Pike 1969). We have not yet completed calculations for this case for finite sampling times and we have presented here final results solely for the unclipped case. When clipping is used these results will have only qualitative significance for high counting rates but will be adequate in the common experimental situation of clipping at zero with low mean. Considerable simplification of the calculation in fact occurs in the low mean limit, when the statistics of the photo-detection process dominate the error. This also occurs in experiments where T_p is so large that the samples of the autocorrelation function are independent. In this case the normalized variance of the single clipped autocorrelation function $G_k^{(2)}(\tau)$ may be shown to take the form ($T \to 0$)

$$\frac{1}{N}\left\{\frac{\bar{n}|g^{(1)}(\tau)|^{-4}}{x^2(1+k)^2}\left(\frac{1+2\bar{n}}{x^{k+1}}-\bar{n}\right)+\frac{|g^{(1)}(\tau)|^{-2}}{x(1+k)}\left(\frac{1+4\bar{n}}{x^{k+1}}-2\bar{n}\right)+\frac{k-2\bar{n}}{(1+k)x^{k+1}}-1\right\} \tag{4}$$

where $x = \bar{n}/(1+\bar{n})$, $\bar{n}$ is the mean number of counts per sample, k is the clipping level and $g^{(1)}(\tau)$ the optical or field autocorrelation function. The expression (4) is applicable to the experiments of Foord *et al.* (1970) in the limit $\bar{n} \ll 1$. The ratio of

the normalized variances of the clipped and unclipped intensity correlation functions then reduces to

$$\frac{1+k+|g^{(1)}(\tau)|^{-2}}{1+|g^{(1)}(\tau)|^{-2}}\frac{1}{\bar{n}^k(1+k)^2} \tag{5}$$

and takes the value unity when $k = 0$. It is interesting to note that the expression (4) is not minimized by clipping at the mean. For example when τ is large so that the first term dominates, the minimum value, as a function of k, occurs when $\bar{n}$ is approximately equal to the expression $1/[\exp\{2/(1+k)\}-1]$.

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